

ON CERTAIN TWO-STEP NILPOTENT LIE ALGEBRAS

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Abstract

In [10], as an application of results from geometric invariant theory, it is shown that a two-step nilpotent Lie algebra associated with a graph is an Einstein nilradical, if and only if the graph is positive. In this note, we give a more elementary proof of this result using the notion of pre-Einstein derivation and some of its properties developed in [11].

1. Introduction

In this note, we are interested in homogeneous Einstein manifolds of negative scalar curvature. The classical examples of Einstein metrics of negative scalar curvature are the symmetric spaces of non-compact type. Other examples are known, see [1], [5], [6], [8], [10], [13]. All known examples of homogeneous Einstein manifolds of negative scalar curvature are isometric to Einstein Riemannian solvmanifolds, e.g., a simply connected solvable Lie group S together with a left invariant Einstein

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Riemannian metric g . In these examples, such a solvable Lie group S is a semi-direct product of an Abelian group A with a nilpotent normal subgroup N , the nilradical. We identify a left invariant Riemannian metric on an arbitrary Lie group S with the inner product \langle, \rangle determined on the Lie algebra \mathfrak{s} of S , and the pair $(\mathfrak{s}, \langle, \rangle)$ will be referred to as a *metric* Lie algebra. If S is Einstein, we say that \mathfrak{s} is an Einstein metric Lie algebra. A nilpotent Lie algebra $(\mathfrak{n}, [,])$ is said to be an Einstein nilradical, if it admits an inner product \langle, \rangle such that there exists an Einstein metric solvable extension of $(\mathfrak{n}, [,], \langle, \rangle)$, that is, an Einstein metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, [,], \langle, \rangle)$, where \mathfrak{a} is Abelian (this fact follows by [9]) and $\mathfrak{a} \perp \mathfrak{n}$ such that the restrictions of the Lie bracket and the inner product of \mathfrak{s} to \mathfrak{n} coincide with the Lie bracket and the inner product of \mathfrak{n} , respectively. We use the same notations $[,]$ and \langle, \rangle for \mathfrak{s} . We note that in the rank-one case, that is, $\dim \mathfrak{a} = 1$, such an Einstein extension, if it exists, is unique. So, rank-one Einstein solvmanifolds are completely determined by its metric nilpotent part. Recall that the study of Einstein solvmanifolds reduces to the rank-one case by [6].

In [2], two-step nilpotent Lie algebras attached to graphs are introduced, where the authors determined their group of Lie automorphisms. In [3] and [4], we considered a *metric* two-step nilpotent Lie algebra $(\mathfrak{n}, [,], \langle, \rangle)$ attached to a graph and studied, whether there exists a rank-one Einstein metric solvable extension of it. We described necessary and sufficient conditions of this problem in terms of the graph.

Recently in [10], among other results, the authors studied a more general problem in the same context. In fact, they considered a two-step nilpotent Lie algebra $(\mathfrak{n}, [,])$ attached to a graph and studied, whether it is an Einstein nilradical. Using results from geometric invariant theory, the authors proved the following result.

Theorem 1. *A two-step nilpotent Lie algebra associated with a graph is an Einstein nilradical, if and only if the graph is positive.*

The positivity of a graph means that a certain uniquely defined weighting on the set of edges is positive ([10]). The exact definition will be given later. In this note, with the same method as in [3], we give a more elementary proof of the above result using the notion of pre-Einstein derivation and some of its properties developed in [11], where an even more general result concerning nilpotent Lie algebras having a nice basis is proved. We recall the construction with graphs introduced in [2] and some notations in the next section.

2. Preliminaries

Let G be a finite graph. We denote by V and E , the set of vertices and edges, respectively. Equivalently E is a collection of unordered pairs of distinct vertices; the unordered pairs will be written in the form vv' , where $v, v' \in V$. In this case, we say that v and v' are joined and the edge vv' has v and v' as its vertices. Let \mathfrak{v} be a vector space with V as a basis. Let \mathfrak{z} be the subspace of $\wedge^2 \mathfrak{v}$, the second exterior power of \mathfrak{v} , spanned by $\{v \wedge v' : v, v' \in V, vv' \in E\}$. Let $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Stipulating the conditions that for any $v, v' \in V$, $[v, v'] = v \wedge v'$, if $vv' \in E$ and 0 otherwise, and $[Z, X] = 0$ for all $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ determines a unique Lie algebra structure on \mathfrak{n} ([2]). Clearly, $(\mathfrak{n}, [\cdot, \cdot])$ is a two-step nilpotent Lie algebra. The center of $(\mathfrak{n}, [\cdot, \cdot])$ is $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$, if and only if the graph has no any isolated vertex. We may assume that the graph has no isolated vertices since such vertices only determine an Abelian factor of $(\mathfrak{n}, [\cdot, \cdot])$. Suppose that $V = \{X_1, \dots, X_k\}$ and $E = \{Z_1, \dots, Z_r\}$, where each $Z_i, 1 \leq i \leq r$ is an edge $X_j X_{j'}$ for some j, j' with $1 \leq j < j' \leq k$. These sets are two bases of \mathfrak{v} and \mathfrak{z} , respectively.

Suppose that $\langle \cdot, \cdot \rangle$ is an arbitrary inner product on \mathfrak{n} . We can then define a linear map $J : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$ by

$$\langle J(Z)X, Y \rangle = \langle [X, Y], Z \rangle,$$

for $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$.

Consider now a metric solvable extension $(\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{n}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ of \mathfrak{n} , where the norm of A is one. So $A = H/|H|$, where H is the unique vector with the property $\langle H, X \rangle = \text{tr}(\text{ad}_X)$ for all $X \in \mathfrak{s}$. We let $f := \text{ad}_A : \mathfrak{n} \rightarrow \mathfrak{n}$, and we can suppose that $\text{tr} f > 0$. Note that $\text{tr}(f) = |H|$. According to [6], if the associated Riemannian solvmanifold is Einstein, then without loss of generality, we can assume that f is symmetric relative to the inner product $\langle \cdot, \cdot \rangle$. In this case, the matrix representation of f in an orthonormal basis of \mathfrak{n} is in the following form ([6] and [7]):

$$f = \begin{pmatrix} B_r & 0 \\ 0 & D_k \end{pmatrix}.$$

In fact, the solvmanifold is Einstein, if and only if there exists a negative constant μ such that we have the following relations:

$$(E_1) \quad -\text{tr}(f)B + \frac{1}{4}J^*J = \mu \text{Id}_r,$$

$$(E_2) \quad -\text{tr}(f)D + \frac{1}{2} \sum_{i=1}^r J^2(z_i) = \mu \text{Id}_k,$$

$$(E_3) \quad J(B(\cdot)) = J(\cdot)D + DJ(\cdot),$$

where $\{z_1, \dots, z_r\}$ is an orthonormal basis of \mathfrak{z} and the endomorphism J^*J of \mathfrak{z} is represented by the $r \times r$ matrix $(-\text{tr}({}^t J(z_i)J(z_j)))_{i,j}$. In the next section, we consider these relations.

3. Result

We prove Theorem 1 in this section. Suppose that $(\mathfrak{n}, [\cdot, \cdot])$ is a two-step nilpotent Lie algebra associated with a graph G as described above. Suppose that $(\mathfrak{n}, [\cdot, \cdot])$ is an Einstein nilradical. Hence, there exists an inner product $\langle \cdot, \cdot \rangle$ on $(\mathfrak{n}, [\cdot, \cdot])$ such that the metric solvable extension $(\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{n}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ is Einstein. We let $f := \text{ad}_A : \mathfrak{n} \rightarrow \mathfrak{n}$ and suppose that $\text{tr} f > 0$. As we mentioned above, without loss of generality, we can assume that f is symmetric relative to the inner product $\langle \cdot, \cdot \rangle$. Let

$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ as above. We may assume that the graph has no isolated vertices, so the center of $(\mathfrak{n}, [,])$ is $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$.

We must show that the graph G is positive. The positivity of G is defined as follows. Recall from graph theory that two distinct edges $Z_i, Z_{i'}$ of the graph G are called *adjacent*, if they share a vertex. The line graph $L(G)$ is the graph, whose set of vertices is E and two of them are joined, if and only if they are adjacent. The adjacency matrix of the graph G with vertex set $V = \{X_1, \dots, X_k\}$ is defined as the symmetric $k \times k$ matrix with a 1 in the entry jj' , if and only if $X_jX_{j'} \in E$ and zero elsewhere. Now, we can define a positive graph as in [10].

Definition. A graph G is said to be *positive*, if the column vector α satisfying

$$(3\text{Id} + \text{Adj}(L(G)))(\alpha) = [1]_r,$$

has all its entries positive, where $\text{Adj}(L(G))$ is the adjacency matrix of the associated line graph $L(G)$ and $[1]_r$ denotes the r -dimensional vector all of whose coordinates are ones.

We denote by $\alpha_1, \dots, \alpha_r$, the coordinates of α . So, we must show $\alpha_i > 0$ for $1 \leq i \leq r$. We had the following basis of \mathfrak{n} : $b = \{X_1, \dots, X_k, Z_1, \dots, Z_r\}$. Construct the $r \times (k + r)$ matrix Y such that the i -th row of Y for $1 \leq i \leq r$ has 1 in the columns j and j' , -1 in the column $k + i$, if and only if $X_jX_{j'} = Z_i$ with $1 \leq j < j' \leq k$ and zero elsewhere. This matrix appears also in [11] and [12], and we will use some ideas of [11] in the sequel as well. Note that the matrix Y can be defined in a more general setting of nilpotent Lie algebras having a nice basis. It is easy to see that with our notations, we have $YY^t = 3\text{Id} + \text{Adj}(L(G))$. Denote the coordinates of the vector $Y^t\alpha$ by $\phi_1, \dots, \phi_{k+r}$. Clearly, $\phi_{k+i} = -\alpha_i$ for $1 \leq i \leq r$. Following [11], with respect to the basis b , define the diagonal operator ϕ on \mathfrak{n} by $\phi = \text{diag}(1 - \phi_1, \dots, 1 - \phi_{k+r})$. It is easy to verify that ϕ is a derivation of \mathfrak{n} . In fact, a diagonal operator (with respect to the basis b) is a derivation of \mathfrak{n} , if and only if the vector consisting of the

diagonal entries belongs to $\ker Y$. A straightforward calculation shows that ϕ has the following property:

$$\mathrm{tr}(\phi \circ \psi) = \mathrm{tr}(\psi),$$

for all diagonal derivations ψ of \mathfrak{n} (with respect to the basis b). Now consider an arbitrary derivation ψ of \mathfrak{n} . With respect to the basis b , construct the diagonal operator ψ' with the same diagonal entries as ψ . Because of the nice structure of the basis b , it is again easy to verify that ψ' is itself a derivation of \mathfrak{n} . As $\mathrm{tr}(\psi) = \mathrm{tr}(\psi')$ and $\mathrm{tr}(\phi \circ \psi) = \mathrm{tr}(\phi \circ \psi')$, we obtain the following ([11]).

Lemma 1. *For any derivation ψ of \mathfrak{n} , we have*

$$\mathrm{tr}(\phi \circ \psi) = \mathrm{tr}(\psi).$$

All of this has been observed in [11], where such a derivation ϕ is called a *pre-Einstein* derivation. Recall from [6] that the operator ad_H has the following property: for any derivation ψ of \mathfrak{n} , we have $\mathrm{tr}(\mathrm{ad}_H \circ \psi) = -\mu \mathrm{tr}(\psi)$, where μ is the negative Einstein constant and this property characterizes the symmetric derivation ad_H up to automorphism of \mathfrak{n} . According to [11], this implies that without loss of generality, we can assume $\mathrm{ad}_H = -\mu\phi$. So, we can assume that $f = \frac{-\mu}{|H|} \phi = \frac{-\mu}{\mathrm{tr}(f)} \phi$. Hence, with respect to the inner product $\langle \cdot, \cdot \rangle$, the eigenspaces of ϕ are orthogonal. In particular, the spaces \mathfrak{v} and \mathfrak{z} are orthogonal. Denote the matrix representation of f in orthonormal bases of \mathfrak{z} and \mathfrak{v} by the following

$$f = \begin{pmatrix} B_r & 0 \\ 0 & D_k \end{pmatrix}.$$

On the space \mathfrak{z} , we had $\phi = \mathrm{diag}(1 + \alpha_1, \dots, 1 + \alpha_r)$ (with respect to the basis $\{Z_1, \dots, Z_r\}$ of \mathfrak{z}). Hence, the eigenvalues of B are the real numbers $\frac{-\mu}{\mathrm{tr}(f)}(1 + \alpha_1), \dots, \frac{-\mu}{\mathrm{tr}(f)}(1 + \alpha_r)$. From the relation (E_1) , we

have $-\operatorname{tr}(f)B + \frac{1}{4}J^*J = \mu\operatorname{Id}_r$, where the endomorphism J^*J of \mathfrak{z} is represented by the $r \times r$ matrix $(-\operatorname{tr}({}^tJ(z_i)J(z_j)))_{i,j}$ with $\{z_1, \dots, z_r\}$, an orthonormal basis of \mathfrak{z} . Choose this orthonormal basis such that J^*J is diagonal. As $J(z)$ is antisymmetric for all $z \in \mathfrak{z}$, we obtain that the eigenvalues of J^*J are all positive numbers. Using the relation (E₁), we have $J^*J = 4(\mu\operatorname{Id}_r + \operatorname{tr}(f)B)$, and we obtain $4(\mu + (-\mu)(1 + \alpha_i)) > 0$ for all $i, 1 \leq i \leq r$. This shows that $\alpha_i > 0$ for all i . So, the graph G is positive. Hence, if a two-step nilpotent Lie algebra associated with a graph is an Einstein nilradical, then the graph is positive and we have proved the “only if” part of Theorem 1.

Conversely, suppose that the graph has no isolated vertices and is positive. Consider the vector α in the definition of positivity with coordinates $(\alpha_1, \dots, \alpha_r)$. The two-step nilpotent Lie algebra $(\mathfrak{n}, [\cdot, \cdot])$ associated to the graph is clearly isomorphic to the vector space \mathfrak{n} with the new Lie bracket $[\cdot, \cdot]'$ defined by $[X_j, X_{j'}]' = \sqrt{\alpha_i}Z_i$, if $Z_i = X_jX_{j'}$ is an edge of the graph with $1 \leq j < j' \leq k, 1 \leq i \leq r$, and zero otherwise, ([10]). Let $\langle \cdot, \cdot \rangle$ be the inner product on \mathfrak{n} relative to which the basis $\{X_1, \dots, X_k, Z_1, \dots, Z_r\}$ is orthonormal. For $(\mathfrak{n}, [\cdot, \cdot]', \langle \cdot, \cdot \rangle)$, we can now construct an Einstein metric solvable extension using the relations (E₁), (E₂), and (E₃) with $\mu = -\frac{1}{2}$. We omit the explicit formulas for the matrices B and D .

Remark. Note that the case $\alpha_1 = \dots = \alpha_r = 1$ leads to the regular line graph, (i.e., all the vertices of the line graph have the same degree). It can be shown that this is equivalent to the case, where each connected component of the graph is regular or a bipartite graph such that all the vertices in each partite set have the same degree. This was the main result in [3].

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